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Optimizing for the Highest Probability of Profit at Some Future Point in Time

A Prospect Theory Implementation for Portfolio Managers

Ralph Vince, February 2008

Abstract

Presented is a small Martingale approach to portfolio allocation, within a given drawdown constraint. Such an approach seeks to maximize the probability of being profitable at some given future point, as opposed to simply seeking to maximize profits. This is in direct contravention to not only mean-variance models, but geometric mean optimization as well. However, maximizing the probability of being profitable at some specified future point is consistent with the requirements imposed on most fund managers. Additionally, the small Martingale operating function presented of quantity at risk to a multiple of an initial stake is representative of the function evolution itself has programmed into simians for risk with respect to a multiple of an initial stake. Thus, the procedure presented is serendipitously consistent with the preferred investor behavior regarding risk aversion posited in "Prospect Theory," and as such, ought to be psychologically easier for an investor or fund manager to implement, and more satisfying to the investor or fund manager's clients.

Background

Often the maximization of the geometric return is given as the primary criterion in portfolio construction and/or bet sizing, Bernoulli (1738), Kelly (1956), Latane and Tuttle (1967). This author as well has provided a paradigm for money management based on geometric mean maximization.

Geometric mean maximization requires small anti-Martingale type progressions. They trade in quantity with respect to account size, increasing trading quantity as equity increases. They are *profit* maximizers, and hence have an equity curve that is mostly flat at best for very long stretches in time, then tends to see enormous and rapid growth such that one can see exponential growth has occurred by the right-hand side of the equity curve.

On the other hand, in maximizing the probability of profit, one is not concerned with geometric growth, nor even smoothness in an equity curve, but rather that at some future point (some "horizon" defined as a designated number of holding periods from the present one) the equity curve is above where it is today, plus some prescribed amount, with the highest probability. This requires a Martingale or small Martingale progression.

Typically, a Martingale doubles the bet size with each losing bet. As soon as the losing streak is broken, a one-unit gain is realized. The downside is that as the losing streak continues, the bet sizes double with each losing play, and eventually the required bet size is unachievable.

Typically then, Martingale and small Martingale-type systems suffer from larger drawdown than their opposing counterparts who trade in quantity relative to account size, such as geometric mean maximization strategies.

This author has previously demonstrated a means of quantifying drawdown, allowing us to now use the constraint of drawdown as our risk metric[1], such that we may now employ a Martingale-style approach within the leverage space terrain – without the corresponding risk of larger drawdown that is usually inherent in such an approach.[2]

This paper attempts to demonstrate a procedure for a small Martingale progression for capitalizing portfolio components which seeks to maximize the probability of being at or above a given return by a specific future point, within the constraint of not exceeding a given probability of touching or exceeding a lower absorbing barrier through the duration towards that future point. This lower barrier may be fixed (i.e. "ruin") or allowed to float upwards as a percentage of equity increase (i.e. "drawdown").

Serendipitously, a small Martingale-type approach is consistent with the preferred investor behavior regarding risk aversion posited in "Prospect Theory," Kahneman and Tversky (1979), and as such, ought to be psychologically easier for an investor or fund manager to implement. In brief, Prospect Theory asserts that humans have a greater tendency to gamble more under accrued losses (i.e. a small Martingale) in an attempt to maximize the probability of profit at a future point in time, whereas those confronted with profits seem to be more risk-averse,[3] save the rare, freelance madman who truly *is* a profit maximizer.

Given the propensity in humans to maximize the probability of profit at a given horizon in time, it becomes the fund manager's responsibility and preference to pursue that within a given drawdown constraint. In one of those rare conjugal visits of Mathematics and human behavior, this paper seeks to identify that function for probability of profit maximization versus risk, as both a tool for the portfolio manager as well as the mathematical operating function of human behavior under conditions of risk.

This paper assumes the reader is already familiar with the Leverage Space Model presented in "The Handbook of Portfolio Mathematics," Vince (2007) and particularly Chapter 12 therein, regarding drawdown as a constraint.

Algorithm & Formulas

If we have a variable, we call a "Martingale exponent," denoted as z greater than -1 and less than or equal to zero:

$$-1 < z <= 0 \qquad (1)$$

We can then say that

$$1/(1+z) - 1 \qquad (2)$$

Gives a result from 0 to infinity, as the following table demonstrates:

$-z$	$1/(1+z)-1$
Approach –1 from the right	Approach infinity
-0.9999	10000
-0.999	1000
-0.9901	100
-0.98039	50
-0.97087	33.33333
-0.96154	25
-0.95238	20
-0.9434	16.66667
-0.8	4
-0.66667	2
-0.5	1
-0.37879	0.609756
-0.04762	0.05
-0.0099	0.01
0	0

Note that since we are going to be trading a small Martingale, we are not trading a fraction of our stake, so there is seemingly no f value for the components and hence no f\$ value.

However, we do need a context, an initial capitalization of a component, a scenario spectrum, and we will retain a consistent nomenclature and call this initial capitalization the component's f\$ (i.e. the amount we divide the current total capital of an account by to know how many 'units' to put on for the current position). And since we have an f\$ for each component, and a scenario comprising the largest losing outcome for each component, we can discern an "initial f" value for each component (as that value wherein the absolute value of the outcome result of the biggest losing scenario divided by equals the f\$).

$$|\text{ Biggest Losing Outcome }| / f = f\$ \qquad (3)$$

Thus
| Biggest Losing Outcome | / *f$* = *f*

Suppose we currently have $120,000 in equity. Further suppose we currently have on 300 shares of a given stock, and we determine that 1 unit is 100 shares. We thus have 3 units on currently – so we can say that our current *f*$ is $40,000, representing the amount we are capitalizing a unit by. Lastly, suppose we know our largest losing scenario assigned to a unit of this stock is $10,000 (per unit, i.e. per 100 shares). Now we can determine *our 'de facto' f* value as (always 0 >= *f* <=1):

| -10,000 | / 40,000 = .25

In an optimal *f* (LSP [6]) -style portfolio, regardless of the individual components, when the portfolio is up, more quantity is traded and vice versa. Similarly now but in reverse we will trade more quantity on the downside for all components, while retaining their ratios to each other. This is the notion of diversifying risk, whereby stronger elements at the time support the weaker ones.

The process now proceeds as follows. At each holding period (i, of q holding periods), for each component (k, of N components), we adjust the *f*$ for the component that period as follows:

$$f\$_{k,i} = \frac{BL_k / -f_k}{\left(\frac{acctEQ_0}{acctEQ_{i-1}}\right)^{\left(\frac{1}{(1+z)}-1\right)}} \quad (4)$$

Where:

$f\$_{k,i}$ = The amount to allocate to the k'th component on the i'th holding period.
BL_k = The largest losing scenario outcome for the k'th component, a negative number.
f_k= The initial *f* value (0<=*f*<=1) for the k'th component, based on its initial capitalization
$acctEQ_0$ = The account equity before the first holding period (i.e. the initial equity).
$acctEQ_{x-1}$ = The account equity immediately before the current period.
z = The "Martingale exponent," value from (1).

We will employ two separate z values, so that our function is consistent with that of Prospect Theory, which demonstrated empirically people's different risk preferences when they were "up," from a given reference point as opposed to "down," from the given reference point.

Thus, we have a z value for when our stake is down from its starting value (i.e. a multiple on the starting stake < 1) and a different z value when the multiple on the starting stake > 1. We will call these values z_- and z_+ respectively.

Note in (4) at $z = 0$ the investor's capitalization per unit remains constant (hence the investor is still trading less as his equity is diminishing). We can show these relationships graphically, and we will assume our initial *f*\$ (when multiple = 1) is \$1. This is demonstrated in Figure 1:

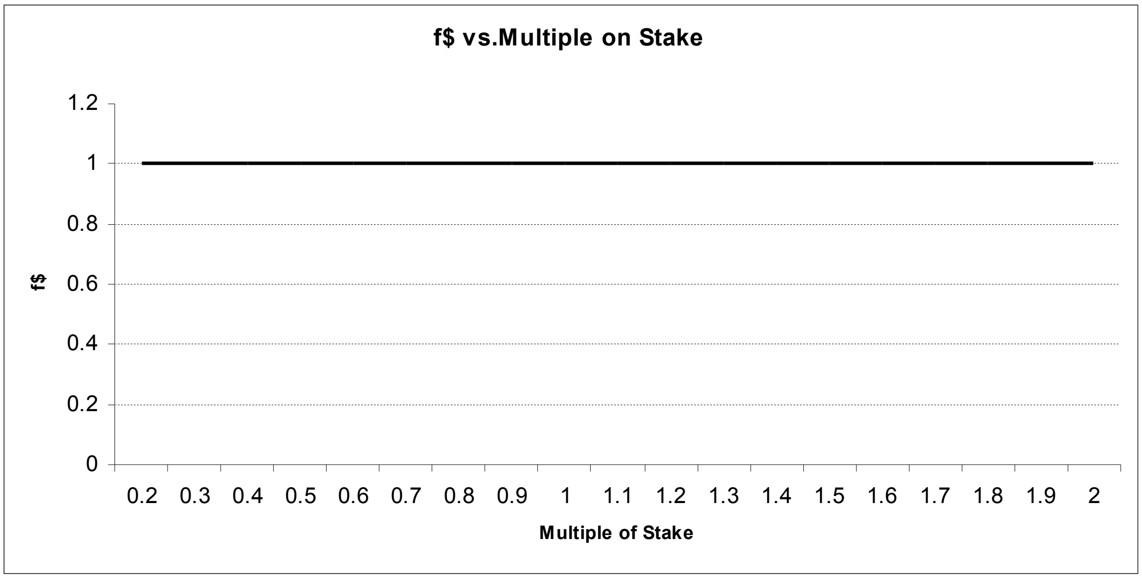

Figure 1
z_-=0, z_+=0

Thus, Figure 1 demonstrates that regardless of equity, at z=0 the number of units put on will always be the current equity divided by the same, initial *f*\$ amount. So as the account equity diminishes, so too will the units put on and vice versa.

At z= -.5 the capitalization is such that the number of units the investor trades in is constant. Thus, at z=-.5 the number of units he trades will always be the same regardless of account equity. This is demonstrated in figure 2:

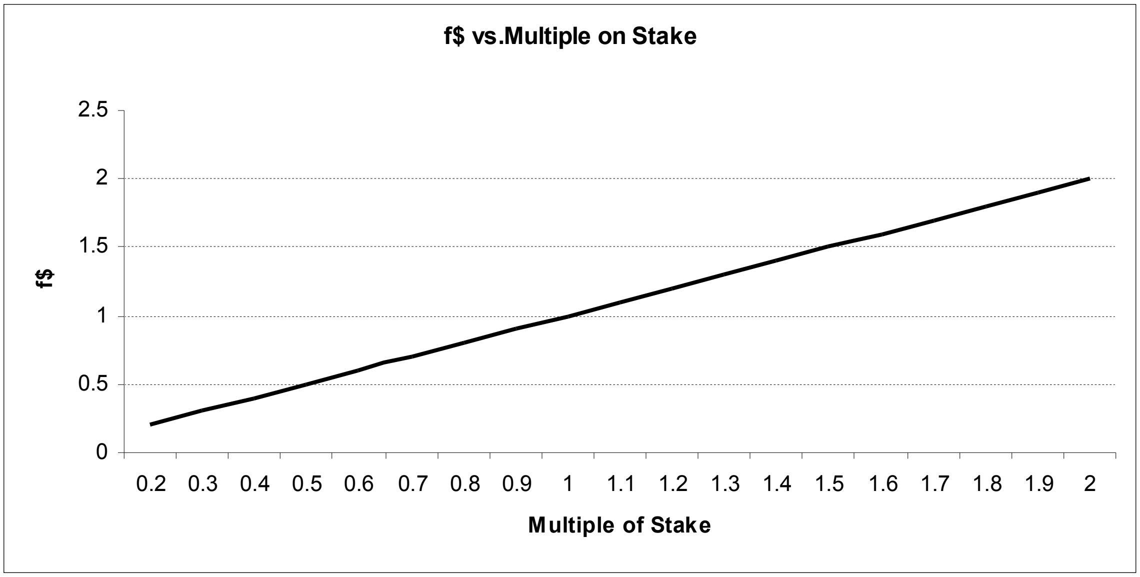

Figure 2
$z_- = -.5$, $z_+ = -.5$

When z <-.5, the investor begins to capitalize units with ever less amounts, thus incurring a Martingale-type effect. Again, since preferences among human beings seem to be a function of whether one is up or down from a given reference point (i.e. where multiple =1), we allow two separate z values to accommodate this very human propensity, though, as the previous two figures demonstrated, these two values could be set equal to each other.

Figure 3 shows (4) in a typical, real life example of two separate z values:

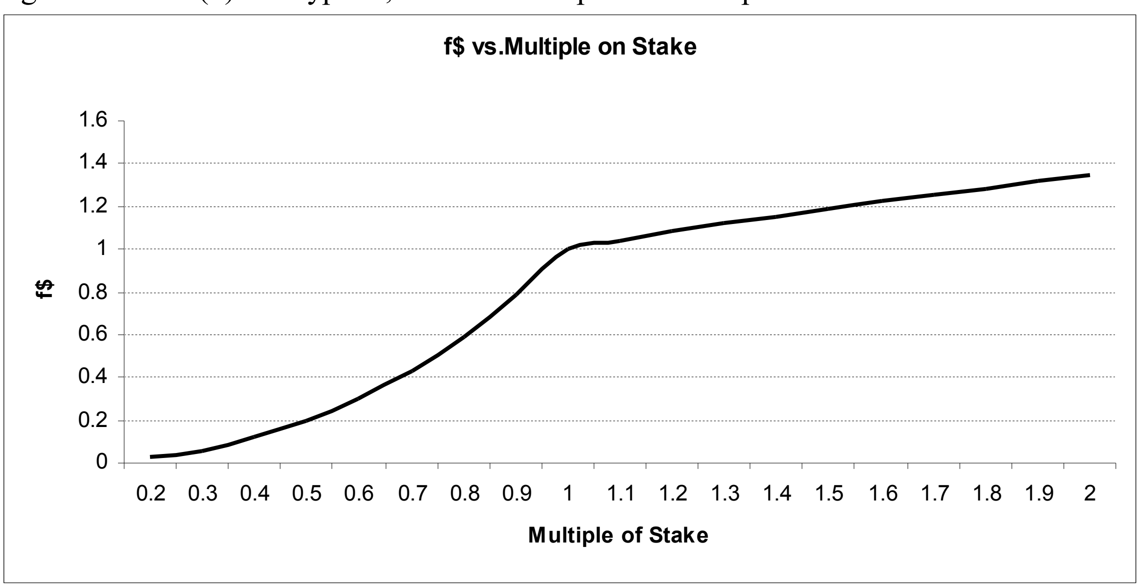

Figure 3
$z_- = -.7$, $z_+ = -.3$

Figure 3 begins to accomplish the small Martingale effect, innate in Prospect Theory, which seeks to maximize profits at a given future point. Figure 3 is shown in tabular form here as:

Multiple	*f$*
0.1	0.004641589
0.2	0.023392142
0.3	0.060248966
0.4	0.117889008
0.5	0.198425131
0.6	0.30363576
0.7	0.435072961
0.8	0.594123371
0.9	0.782046402
1	1
1.1	1.041692943
1.2	1.081271659
1.3	1.11900716
1.4	1.155117866
1.5	1.189782789
1.6	1.223150769
1.7	1.255347099
1.8	1.28647841
1.9	1.316636313
2	1.345900193

It is this author's contention that (4) is the evolutionary hard-wired function in humans pertaining to risk seeking and risk aversion, and is consistent in graphical form with what has been posited by Prospect Theory [3].

Given that this (4) is the evolutionary-wired preference in humans, a fund manager's "success," becomes a function of what degree he satisfies this preference in his clients.

Note that in a straight Martingale progression (betting 1,2,4,8,16, *ad infinitum* after a loss until a win is seen), only after a winning play is the account value up. For all other situations, the account is actually down most of the time, and often quite substantially if during a run of losing plays. Thus, a straight Martingale can be said to put an account up at some arbitrary point in the future, but not necessarily at some *given* point in the future.

Hence, the fact that the function (4) "breaks," at a multiple = 1, by virtue of having two separate Martingale exponents (z_- and z_+). Note that when an account is up, the bet size diminishes as a function of both the multiple on the starting stake, and the Martingale exponent z_+. This part of the formula, consistent with the empirically observed behavior of Prospect Theory, permits a small Martingale progression the opportunity to retain profits efficiently until the horizon, the *given* future point (as opposed to the *arbitrary* one), is seen.

In (4), since $f\$_{k,i}$ = The amount to allocate to the k'th component on the i'th holding period, we can determine the number of units to trade for the k'th component on the i'th holding period as:

$$U_{k,i} = \frac{acctEQ_{i-1}}{f\$_{k,i}}$$

(5)

Dividing current equity by the $f\$$ given in (4) for each component tells us how many units to have on for each component (1…N) at each holding period (1…q).

Simply put, we will now attempt to find the Martingale exponents (z_- and z_+) for the portfolio, and the set of $f_{1\ldots N}$ for its components that maximize the probability of profit (PP) at some given horizon. Additionally, we can discern this with respect to a given probability of a given drawdown or risk of ruin, at that horizon point, per the technique given in Vince [6], with slight amendments provided at the end of this paper.

Note you trade more units when the account is down, and less when it is up. The Martingale exponents (z_- and z_+) are the levers that govern this. We will then seek those values for the Martingale exponents (z_- and z_+) for the portfolio, and the set of initial $f_{1\ldots N}$ for its components so as to maximize the probability of profit, PP(r) at some given horizon, where r typically is 0. If, say, we considered "profit," as a 2% return at the horizon, we would say we seek to maximize PP(.02).

Note we can still calculate our TWR($f_1 \ldots f_N$) ("Terminal Wealth Relative") as:

$$TWR\,(f_{1\ldots}f_N) = \frac{acctEQ_q}{acctEQ_0}$$

(6)

Since TWR($f_1 \ldots f_N$) is simply the multiple we have made on our stake, after *q* holding periods.

Note a different scenario occurs at each holding period. We wish to maximize the probability of profit, PP(r) for a given r, over a given number of holding periods, *q*.

For the sake of simplicity, assume a coin toss, wherein one of two possible scenarios, Heads or Tails, H or T, can occur. If we decide we are going to look at *q* = 2 holding periods, there are four possible branches that can be traversed, as follows:

H	H
H	T
T	H

T T

Note in this case of two possible scenarios, and $q = 2$ holding periods, there are 4 possible branches of traversal (this is the process detailed in [6], Chapter 12 for branch traversal in determining drawdown probabilities).

Similarly, if we assume a portfolio of two coins, each with the same possible two scenarios of heads and tails, we look at

HH	HH
HH	HT
HH	TH
HH	TT
HT	HH
HT	HT
HT	TH
HT	TT
TH	HH
TH	HT
TH	TH
TH	TT
TT	HH
TT	HT
TT	TH
TT	TT

Each branch sees its own TWR($f_1 \ldots f_N$) calculation from (6). Thus, for each branch, we can determine if:

$$TWR\ (f_{1\ldots} f_N) - 1 \geq r \tag{7}$$

for that branch, and if so, we conclude that branch is "profitable." Thus, since we wish to optimize for highest probability of profit, we wish to mazimize the ratio of the number of branches satisfying (7) divided by the total number of branches – this represents the "Probability of Profit" function, PP(r), which is what we seek to maximize by altering the Martingale exponents (z_- and z_+) for the portfolio, and the set of initial $f_{1\ldots N}$ for its components .

The process for doing this, i.e. finishing at or above an upper absorbing barrier (at time q) is very similar to the process of determining drawdown, i.e. touching or exceeding a lower absorbing barrier (at any time $0\ldots q$). We use the branching process described in [6, Chapter 12] to determine this, yet, in determining the probability of profit we are only concerned if the terminal leaves on the branching process are at or above r, unlike drawdown, where we are concerned if at any point along the branch, b ([6] Chapter 12, as RX(b)), has been touched on the downside.

Thus, at each terminal node in the branch, we assess (7)

The calculation to determine the $acctEQ_y$ at any point (1…q) is given as:

$$acctEQ_y = \sum_{i=1}^{y}\sum_{k=1}^{N}\left(U_{k,i} * outcome_{k,i}\right) \tag{8}$$

Where:

$outcome_{k,i}$ = the k'th component's scenario outcome at point i in the branching process.

Now we can look at the values at the terminal leaf of each branch, and assess (7). Each branch as a probability associated with it, by taking the sum of these probabilities as our denominator, and the sum of those probabilities for those branches that satisfy (7) as our numerator, we derive a PP(r) for a given Martingale exponents (z_- and z_+) and set of initial $f_{1\ldots N}$ values which we are optimizing over to determine greatest PP(r) within an (optional) drawdown / risk of ruin constraint.

Since the process of maximizing probability of profit is an additive one (as opposed to maximizing for profit, which is a multiplicative one), we must amend our calculation for β [6 Chapter 12]. We thus now have, if we are determining risk of ruin, RR(b) as:

$$\operatorname{int}\left(\frac{\sum_{i=1}^{q}(acctEQ_i / acctEQ_0 - b)}{\sum_{i=1}^{q}\left|(acctEQ_i / acctEQ_0 - b)\right|}\right) = \beta$$

(9)

Supplanting [12.03].

Given the propinquity of drawdown and ruin, we must adjust for the case of Risk of Drawdown, RD(b), and supplanting [12.03a] we have:

$$\operatorname{int}\left(\frac{\sum_{i=1}^{q}(acctEQ_i / \max(acctEQ_0 \cdots acctEQ_i) - b)}{\sum_{i=1}^{q}\left|(acctEQ_i / \max(acctEQ_0 \cdots acctEQ_i) - b)\right|}\right) = \beta$$

(9a)

Note that we can perform the calculation for both PP(r) and RX(b) simultaneously in the branching process per the algorithm provided in [6 Chapter 12]. However, it must be pointed out that if the lower absorbing barrier, *b*, is seen while traversing a branch, the branch must still be fully traversed to determine (7). That is, simply hitting a drawdown on the branch does not permit one to abort the branch altogether – the branch must still be fully traversed so as not to sabotage the probability of profit calculation.

Conclusion

The paradigm for examining money management provided by the Leverage Space Model has afforded us an end separate than merely maximizing geometric returns.

It should be pointed out that even though we seem to approach our allocations and leverage (referring to both manifestations of "*leverage" pointed out in [6]*; the immediate snapshot of ratio of quantity to cash, as well as how we progress that ratio through time as equity changes, intimating that leverage, in this second sense, is germane even to a cash account!) from an entirely different standpoint than the multiplicative one innate in geometric mean maximization strategies (and thus innate in the Leverage Space Model), we are still somewhere on the terrain of leverage space, only moving along that terrain as our equity changes; the veracity and relevance of that model is unchanged. Rather, the technique described herein seeks to find a path through that terrain which maximizes the probability of profit within a given drawdown constraint. (In fact, without the paradigm provided by the Leverage Space Model, such an approach would not have been feasible).

People, including fund managers and individual investors, are *not* wealth maximizers. They are maximizers of probability of profit at some given horizon in time, as demonstrated empirically in Prospect Theory, and also further evidenced by the near-universal, visceral reaction exhibited towards the notion of mathematically optimal wealth maximization afforded by geometric mean maximization.

But neither the palette nor evolution itself dictates Mathematics. Regardless of preferences and attitudes towards risk, everyone exists somewhere in the terrain of leverage space at all times. And as a paradigm, the Leverage Space model allows us to trace a path, see the results of our actions, to satisfy the (often seemingly pathological) palate of the individual, such as exhibited by Prospect Theory, which seeks not to maximize wealth, not to find the highest point in the leverage space landscape, but to trace a path through that landscape so as to maximize the probability of profit at a given future point in time.

Further, the Leverage Space Model, since it utilizes the real-world risk metric of drawdown, now permits a small Martingale (demonstrated herein to model the risk preferences of Prospect Theory) to be implemented. We can determine therefrom what our allocations and progressions of those allocations should be, i.e. our "path," through leverage space, so as to accommodate Prospect Theory's *implied* criterion of "maximum probability of profit at a given horizon in time," by determining those parameters that dictate our path.

We now have a method which allows fund managers to select a horizon in the future whereby they can maximize their probability of profit or of a minimum return, within a given drawdown constraint, in the context of the Leverage Space Model itself.

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[1] Though the Leverage Space Model is presented as specifying risk as drawdown rather than variance in returns or some other ersatz measure of risk, it is feasible to incorporate these other risk measures, either in tandem with drawdown, or in solitary fashion, using the Leverage Space Model. For example, if a manager is indexed to a variance-based benchmark, such as the Sharpe Ratio, he *could* employ the Leverage Space Model, paring away those points on the terrain where either the drawdown constraint <u>*or*</u> his variance constraint was violated, thus making points that violate either constraint be unacceptable portfolio combinations.

[2] Recall in the Leverage Space Model if one is trading in a constant-unit size (as opposed to trading in size relative to equity), one is migrating towards the $f_1 \ldots f_n = 0_1 \ldots 0_n$ point in leverage space as the equity increases, and, similarly towards the $f_1 \ldots f_n = 1_1 \ldots 1_n$ point in leverage space as the equity decreases. Since we are always within the terrain of leverage space, whether we acknowledge it or not, the approach presented, i.e. a Martingale-style approach, can be said to be an approach which seeks a path through the terrain of leverage space itself. Hence, we see firsthand here how the Leverage Space Model is not merely a static model of allocation, but a paradigm for more dynamic-types of allocations as well.

[3] The pervasiveness of this tendency in humans and other primates, Chen *et al.* (2006), is suggestive of an evolutionary cause, a hard wiring of a given function.